國立中山大學96學年度博士班招生考試試題

科目：機率論 【應數系甲組】

20 points for each problem

1. Assume the number of accidents occurred yearly in a certain highway follows the Poisson distribution with a random parameter Λ which has the distribution function $\Gamma(a, \mu)$, a, $\mu > 0$.

   (i) Find the unconditional distribution of X.

   (ii) Find the conditional distribution of Λ for given X.

2. Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) with common distribution function (c.d.f.) F, $F(0) = 0$. Let $Z = (X_1 X_2 \ldots X_n)^{1/n}$ be the geometric average of the $X_k's$.

   (i) Find the expression of the characteristic function (ch.f.) of $Y_k = \log X_k$, and the exact form of the ch.f. of Y_k when X_k is uniformly distributed on $(0, 1)$.

   (ii) Find the ch.f. of $-2n\,Z$ and the corresponding distribution function.

3. Consider the random variable Y which has negative binomial distribution with probability mass function

$$f(y; k, \mu) = \frac{\Gamma(y+k)}{\Gamma(k)\Gamma(y+1)} \left(\frac{k}{\mu+k}\right)^k \left(1 - \frac{k}{\mu+k}\right)^y,$$

   $k \geq 1$, $y = 0, 1, 2, \ldots$, where k and μ are parameters.

   (i) Find the expectation and variance of Y, i.e. $E(Y)$ and $\text{Var}(Y)$ in terms of k and μ.

   (ii) Determine the limiting distribution of Y when $k \to \infty$.

4. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Also let F_n, f_n denote the c.d.f. and p.d.f. of X_n respectively.

   (i) State the definition that $\{X_n, n \geq 1\}$ converges in distribution to a random variable X.

   (ii) For $\forall n \geq 1$, let

$$f_n(x) = \begin{cases} 1/2 & , \text{ if } x = 1 - 1/n \text{ or } 1 + 1/n, \\ 0 & , \text{ otherwise.} \end{cases}$$

   Is $\{X_n, n \geq 1\}$ convergent to a random variable X when $n \to \infty$? If it is, what is the distribution of X?

   (iii) Is f_n convergent to a p.d.f when $n \to \infty$? If it is, what is it?

5. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume for all $n \geq 1$, $E(X_n) = \mu$, $\text{Var}(X_n) = \sigma_n^2 \leq b$, where $b > 0$ is a constant.

   (i) State the definition that $\{X_n, n \geq 1\}$ converges in probability to a random variable X.

   (ii) Show that $\overline{X}_n = \sum_{i=1}^n X_i / n$ converges in probability to μ when $n \to \infty$, under the condition that

$$\lim_{|i-j| \to \infty} \text{Cov}(X_i, X_j) = 0.$$

國立中山大學96學年度博士班招生考試試題

科目：**數理統計**　【應數系甲組】

共五題每題20分。答題時，每題都必須寫下題號與詳細步驟。

1. Suppose $X_1, X_2, \ldots$ are jointly continuous and independent, each distributed with marginal pdf $f(x)$, where each X_i represents annual rainfall at a given location.

   (a) Find the distribution of the number of years until the first year's rainfall, X_1, is exceeded for the first time.

   (b) Show that the mean number of years until X_1 is exceeded for the first is infinite.

2. Let $X_1, \ldots, X_{n+1}$ be iid Bernoulli(p), and define the function $h(p)$ by

$$h(p) = P\left(\sum_{i=1}^{n} X_i > X_{n+1} \middle| p\right),$$

   the probability that the first n observations exceed the $(n+1)$st.

   (a) Show that

$$T(X_1, \ldots, X_{n+1}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

   is an unbiased estimator of $h(p)$.

   (b) Find the best unbiased estimator of $h(p)$.

3. Suppose that when the radius of a circle is measured, an error is made that has a $N(0, \sigma^2)$ distribution. If n independent measurements are made, find an unbiased estimator of the area of the circle. Is it best unbiased?

4. Derive the likelihood ratio test of $H_0 : \mu = \mu_0$ vs. $H_0 : \mu \neq \mu_0$ in the case of the parameter μ of the Poisson distribution. Express the test in an equivalent form involving $\bar{X}$.

5. Derive a general two-dimensional confidence region for the vector parameter $\theta = (\mu, \sigma^2)$ in the $N(\mu, \sigma^2)$ distribution. Plot the confidence region.

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科目：數值分析 【應數系乙組】

Entrance Exam for the Ph.D Program of Scientific Computing

Six questions with the marks indicated.

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1. (15) Describe convergence and stability for numerical methods, give relations between them, and provide examples to explain your answers.

2. (15) Suppose that there exists a root of $f(x) = 0$, and $0 < m \le f'(x) \le M$. Prove that

$$x_{n+1} = x_n - \lambda f(x_n)$$

yields the convergent sequence $\{x_n\}$ to the root for arbitrary $x_0 \in (-\infty, \infty)$ and $0 < \lambda < 2/M$.

3. 15) Let

$$A \in R^{n\times n},\ Cond.(A) = \{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}\}^{1/2},$$

where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximal and the minimal eigenvalues of matrix A, respectively. Prove
(1). $Cond.(AB) \le Cond.(A)Cond.(B)$,
(2) $Cond.(UA) = Cond(A)$, where $U \in R^{n\times n}$ is an orthogonal matrix.

4. (15) Describe the singular value decomposition for the matrix $A \in R^{m\times n}, m \ge n$, and prove that the singular values are non-negative.

5. (20) Derive the error bounds for the trapezoidal rule in two dimensions:

$$\int_0^h \int_0^k g(x,y)dxdy \approx \frac{hk}{4}(g(0,0) + g(h,0) + g(0,k) + g(h,k)),$$

where the function g is smooth enough.

6. (20) Consider the system of ordinary differential equations,

$$\frac{d\vec{u}}{dt} + A\vec{u} = \vec{f},$$

where $\vec{f}$ is known and $\vec{u}(0)$ is given. The matrix A is symmetric and positive definite. Provide truncation errors, and derive stability analysis for the following scheme:

$$\frac{w^{n+1} - w^n}{\Delta t} + \frac{1}{2}A\{w^{n+1} + w^n\} = \vec{f}^{n+\frac{1}{2}}, \quad n \ge 0,$$

where w^n is used to approximate $\vec{u}$ at $n\Delta t$.

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科目：分析　【應數系丙組選考】

In the following, $\mathbb{R}^k$ denotes the Euclidean space of dimension $k > 0$, and write $\mathbb{R}^1 = \mathbb{R}$. For any given Lebesgue measurable subset E of $\mathbb{R}^k$, let $m(E)$ denote its Lebesgue measure.

1. Let $I = [0, 1]$, and let $f : I \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{1}{n} & \text{whenever } 2^{-n} < x \le 2^{-n+1} \text{ and } n > 0 \text{ is an integer,} \\ 0 & \text{whenever } x = 0. \end{cases}$$

Prove that f is Riemann integrable on I, and that $\int_0^1 f(x)\,dx = \ln 2$, where ln denotes the natural logarithmic function. (20 %)

2. Let I be an open interval in $\mathbb{R}$, and let $f : I \longrightarrow \mathbb{R}$ be a continuous function. Assume that

$$f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2} \quad \text{for all } x,\, y \in I.$$

Prove that $f((1-t)x+ty) \le (1-t)f(x)+tf(y)$ for all real numbers t with $0 < t < 1$ and for all $x,\, y \in I$. (20 %)

3. Let D be a nonempty subset of $\mathbb{R}^k$, and let $f : D \longrightarrow \mathbb{R}$ be a function. Assume that for every $r \in \mathbb{R}$ there is an open set $U_r \subset \mathbb{R}^k$ such that $U_r \cap D = \{x \in D : f(x) < r\}$.

(a) Prove that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in D with $\lim_{n\to\infty} x_n = x \in D$, then $\limsup_{n\to\infty} f(x_n) \le f(x)$. (10 %)

(b) Prove that if D is compact, then there exists $x_0 \in D$ such that $f(x_0) \ge f(x)$ for all $x \in D$. (10 %)

4. Let $I = \{x \in \mathbb{R} : 0 \le x \le 1\}$, and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function such that

(i) the first partial derivative $\frac{\partial f}{\partial x}$ exists and is bounded on $\mathbb{R} \times I \subset \mathbb{R}^2$, and

(ii) for every $x \in \mathbb{R}$ the function $y \longmapsto f(x,\, y)$ is Lebesgue integrable on I.

Prove that for every $x \in \mathbb{R}$ the function $y \mapsto \frac{\partial f}{\partial x}(x,\, y)$ is Lebesgue measurable on I, and that $\frac{d}{dx}\int_0^1 f(x,\, y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x,\, y)\,dy$ for $x \in \mathbb{R}$. (20 %)

5. For every integer $n > 0$, let $f_n : \Omega \longrightarrow \mathbb{R}$ be a Lebesgue measurable function, where Ω is a Lebesgue measurable subset of $\mathbb{R}^k$ with $m(\Omega) < \infty$. Assume that for every $x \in \Omega$ there is a real number $\rho_x > 0$ such that $|f_n(x)| \le \rho_x$ for all n. Prove that for any given $\varepsilon > 0$ there exist a real number $\rho > 0$ and a closed subset E of $\mathbb{R}^k$ such that $E \subset \Omega$, $m(\Omega - E) < \varepsilon$ and $|f_n(x)| \le \rho$ for all integers $n > 0$ and all $x \in E$, where $\Omega - E = \{x : x \in \Omega \text{ and } x \notin E\}$. (20 %)

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科目：組合數學　【應數系丙組選考】

Qualifying Examination in Graph Theory

1. (10 points) Let G be a $2k$-connected graph. Suppose $e_1, e_2, \cdots, e_k$ are vertex disjoint edges of G and v is a vertex of G. Prove that G has k cycles $C_1, C_2, \cdots, C_k$ such that C_i contains v and e_i, and moreover for $i \neq j$, C_i and C_j are vertex disjoint except that they both contain v.

2. (16 points) A graph G is k-choosable if for any mapping L which assigns to each vertex v of G a set $L(v)$ of k colours, there is a proper colouring c of G with $c(v) \in L(v)$ for every $v \in V(G)$. Prove that the complete k-partite graph $K_{2,2,\cdots,2}$ with each partite set of cardinality 2 is k-choosable.

4. (16 points) Suppose G is a cubic graph and has a 3-edge coloring f with colours $1, 2, 3$. Prove that for any vertex set $X \subseteq V(G)$, for any $i \in \{1, 2, 3\}$, $|E[X, \bar{X}] \cap (f^{-1}(i)|$ has the same parity as $|E[X, \bar{X}]|$.

5. (16 points) Prove that a graph G has a nowhere zero k-flow if and only if there is an orientation of G such that for any subset X of $V(G)$, at least $1/k$ of the edges in $E[X, \bar{X}]$ are directed from X to $\bar{X}$.

6. (16 points) A homomorphism of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. Suppose there is a homomorphism of an n-vertex graph G to H and H is vertex transitive. Prove that $\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|$.

7. (16 points) Suppose $\chi(G) = k$ and any proper induced subgraph has chromatic number at most $k-1$. Show that G is $(k-1)$-edge connected.

8. (10 points) Prove that if G is a planar graph with n vertices, $n+k$ edges, then G has a cycle of length at most $2(n+k)/(k+2)$.