國立中山大學95學年度博士班招生考試試題

科目：機率論【應數系甲組】

(1) Let $X_1, \dots, X_n$ be independent and identically distributed (iid) with continuous probability density function (p.d.f.) $f(x)$ and cumulative distribution function (C.D.F.) $F(x)$. Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the order statistics of $\{X_i, 1 \le i \le n\}$.(15pts)

(i) Find the joint p.d.f. of $(F(X_{(1)}), \cdots, F(X_{(n)}))$.

(ii) Find $\forall 2 \le k \le n$,

$$P(F(X_{(k)}) - F(X_{(k-1)}) > t), \forall 0 \le t \le 1.$$

(2) Let $X_1, X_2, \cdots, X_n$ be iid $\chi^2(1)$ randomvariables. Let $Z_n = (S_n - n)/\sqrt{2n}$, where $S_n = \sum_{k=1}^{n} X_k$. Use Taylor's approximation of the moment generating function of Z_n to find the asymptotic distribution of Z_n.(15pts)

(3) Let U, V be two r.v.'s, where U has Rayleigh distribution with p.d.f.

$$f_U(u) = \begin{cases} \sigma^{-2} u e^{-u^2/2\sigma^2} & , u \ge 0, \\ 0 & , u < 0, \end{cases}$$

and V is uniformly distributed with $\mathcal{U}(-\pi, \pi)$ distribution. Prove that $X = U\cos V$ and $Y = U\sin V$ are two independent r.v.'s, and find their distributions.(20pts)

(4) Let $X_1, X_2, \cdots,$ be iid random variables with common distribution. Show that $n^{-1}S_n$ converges to a finite constant μ almost surely, if and only if $E|X| < \infty$ and $\mu = E(X)$.(20pts)

(5) Let Z be a r.v. with continuous p.d.f. f, where $f(z) > 0, \forall z \in R$. Find $P(Z > 0 | |Z| = y)$, $y > 0$.(15pts)

(6) Let $X_1, X_2, \cdots$ be iid N(0,1) random variables. Determine the **limiting distribution** of the random variable (15pts)

$$W_n = \sqrt{n}\frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2}.$$

科目：數理統計【應數系甲組】

共五題每題20分。答題時，每題都必須寫下題號與詳細步驟。

1. Let $(X_1, \ldots, X_n)$ have a multinomial distribution with m trials and cell probabilities $p_1, \ldots, p_n$. Show that, for every i and j,

$$X_i | X_j = x_j \sim \text{binomial}\left(m - x_j, \frac{p_i}{1 - p_j}\right)$$

$$X_j \sim \text{binomial}(m, p_j)$$

and that $\text{Cov}(X_i, Y_j) = -mp_ip_j$.

2. If X has pdf $f_X(x)$ and Y, independent of X, has pdf $f_Y(y)$, establish formulas for the pdf of the random variable Z in each of the following situations.

   (a) $Z = X - Y$

   (b) $Z = XY$

3. Let $X_1, \ldots, X_n$ be iid Poisson(λ), and let λ have a gamma(α, β) distribution.

   (a) Find the posterior distribution of λ.

   (b) Calculate the posterior mean and variance.

4. Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1 + e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Show that this family has an MLR.

   (b) Based on one observation, X, find the most powerful size α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. For $\alpha = .2$, find the size of the Type II Error.

5. Find a $1 - \alpha$ confidence interval for θ, given $X_1, \ldots, X_n$ iid with pdf

   (a) $f(x|\theta) = 1$, $\theta - \frac{1}{2} < x < \theta + \frac{1}{2}$.

   (b) $f(x|\theta) = 2x/\theta^2$, $0 < x < \theta, \theta > 0$.

～全卷完～

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科目：分析【應數系丙組選考】

Throughout the exam, $m(E)$, $E \subseteq \mathbb{R}$ denotes the Lebesgue measure of E, and $\int_E f$ means the Lebesgue integral of f.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that f is an open mapping if and only if f is monotonic. (15pt)

2. Let $\varphi : [0,1] \to \mathbb{R}$. Show that $\varphi \in L^\infty[0,1]$ if and only if $\varphi f \in L^1[0,1]$ for all $f \in L^1[0,1]$. (15pt)(Hint: For the "if" part, deny!)

3. A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure on $E \subseteq \mathbb{R}$ if

$$\lim_{n\to\infty} m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} = 0 \text{ for every } \varepsilon > 0.$$

Now let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ such that

$$\int_0^1 |f_n|^\alpha \longrightarrow 0$$

as $n \to \infty$ for some $\alpha > 0$. Show that $f_n \to 0$ in measure. (15pt)

4. We say that $\{f_k : f_k : [0,1] \to \mathbb{R}_0^+, \forall k\}$ is uniformly integrable (UI) if

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \int_0^1 f_k \mathbf{1}_{\{f_k>n\}} = 0,$$

where $\mathbf{1}_E$ is the characteristic function on E. Show that

a. $\sup_k \int_0^1 f_k < \infty$ and $\lim_{n\to\infty} \sup_k m\{f_k > n\} = 0$ if $\{f_k\}$ is UI

b. The UI property is closed under addition.

c. If $f_k \to f$ a.e. and $\{|f_k|\}$ is UI, then $\lim_{k\to\infty} \int_0^1 f_k = \int_0^1 f$ and $\lim_{k\to\infty} \int_0^1 |f_k - f| = 0$.

d. If $|f_k| \leq g$ a.e. for some $g \in L^1$ then $\{f_k\}$ is UI.

e. Is the converse of (d) true? (20pt)

5. A function f on $E \subseteq \mathbb{R}$ is said to satisfy the Lipchitz condition with exponent γ if

$$|f(x) - f(y)| \leq C|x-y|^\gamma,\ x, y \in E$$

for some $C > 0$. Let ϕ be the Cantor-Lebesgue function on $[0,1]$.

a. Show that ϕ can be approximated by a sequence of functions $\{f_n\}$ satisfying the Lipchitz condition with exponent $\gamma = \log 2/\log 3$.

b. Show that ϕ satisfies the Lipchitz condition with exponent $\gamma = \log 2/\log 3$. (20pt)

6. Let f be an entire function on $\mathbb{C}$. Show that either f is constant or $f(\mathbb{C})$ is dense in $\mathbb{C}$. (Hint: Assume that f is not a polynomial. Now consider $g(z) = f(1/z)$ and use the Casoratti-Weierstrass theorem: If z_0 is an essential singularity of an analytic function g, then $g(\{z : |z - z_0| < \varepsilon,\ z \neq z_0\})$ is dense for all $\varepsilon > 0$) (15pt).

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科目：組合數學【應數系丙組選考】

1 (14 points). Suppose G is a 2-connected graph which is not a triangle. Prove that for any edge e of G, $G-e$ is 2-connected or G/e is 2-connected. (Here G/e is the graph obtained from G by contracting edge e.)

2 (14 points). Suppose $G=(V,E)$ is a simple graph, and that for every subset A of V we have $|N(A)| \geq |A|$, where $N(A)=\{y \in V : \exists x \in A, (x,y) \in E\}$. Prove there is a multi-subset E' of E (i.e., an element of E can occur more than once in E') such that each vertex of G is incident to exactly two edges of E' (one edge occurring twice in E' is counted as two edges).

3 (14 points). Suppose P is a partial ordered set. A chain of P is a set C of elements such that any two elements of C are comparable. A antichain of P is a set B of elements such that any two elements of B are not comparable. Prove that if the maximum antichain of P has cardinality k then P can be decomposed into the union of k chains.

4 (14 points). Suppose G is a connected graph of maximum degree $k \geq 3$ on $n \geq k+2$ vertices. Prove that $\alpha(G)\lceil n/k \rceil$, here (and in Problem 6) $\alpha(G)$ is the cardinality of a maximum independent set of G.

5 (14 points). Let $A_1, A_2, \cdots, A_n$ be n subsets of $\{1,2,\cdots,n\}$. Prove that there is an element $i \in \{1,2,\cdots,n\}$ such that for any $j \neq j'$, $A_j - \{i\} \neq A_{j'} - \{i\}$.

6 (15 points). A homomorphism from a graph G to a graph G' is a mapping $f: V(G) \to V(G')$ such that $f(x)f(y) \in E(G')$ whenever $xy \in E(G)$. Prove that if there is a homomorphism from G to G' and G' is vertex transitive then $\alpha(G) \geq \alpha(G')|V(G)|/|V(G')|$.

7 (15 points). Suppose G is a triangle free planar graph. Prove that there is a homomorphism from G to a triangle free planar graph H in which each facial cycle has length at least 5.