國立中山大學九十三學年度博士班招生考試試題

科目：數理統計【應數系甲組】

共九題，第六題20分，其餘每題10分。答題時，每題都必須寫下題號與詳細步驟。

1. Show that if X is a continuous random variable, then

$$\min_a E|X-a| = E|X-m|,$$

where m is the median of X.

2. Let X be a random variable with moment-generating function $M_X(t)$, $-h < t < h$. Prove that $P(X \geq a) \leq e^{-at}M_X(t)$, $0 < t < h$.

3. Find the pdf of $\prod_{i=1}^{n} X_i$, where the X_is are independent uniform$(0,1)$ random variables.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with a pdf

$$f_X(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

5. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. Find a function of S^2, the sample variance, say $g(S^2)$, that satisfies $E[g(S^2)] = \sigma$.

6. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = e^{-(x-\mu)}$, where $-\infty < \mu < x < \infty$.

   (a) Show that $X_{(1)} = \min_i X_i$ is complete sufficient statistic.

   (b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent.

7. Let $X_1, \ldots, X_n$ be a sample from the *inverse Gaussian* pdf,

$$f(x|\mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\{-\lambda(x-\mu)^2/(2\mu^2 x)\}, \quad x > 0$$

Find the MLEs of μ and λ.

8. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential(μ). Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

9. Derive a confidence interval of p for a random sample $X_1, \ldots, X_n$ from Bernoulli(p) by inverting the LRT of $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

～全卷完～

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科目：機率論【應數系甲組】

(20 points for each problem)

(1) Assume that Y is an L^p random variable, that is $E(|Y|^p) < \infty$, for some $p > 0$, and $\{X_n\}$ is a sequence of random variables such that $P(|X_n| \le Y) = 1$, for all $n \ge 1$. Prove that if X_n converges to 0 in probability, then X_n converges to 0 in L^p.

(2) If $E(Z) = 0, E(Z^2) = v^2$ and $E(Z^4) = k^4 > 0$, then $P(Z \ge 0) \ge \frac{v^4}{4k^4}$.

(3) Let $X_1, X_2, \cdots, X_n$ be independent random variables with common mean 0 and variance $\sigma_k^2, k = 1, 2, \cdots, n$, respectively. Prove that for any $\epsilon > 0$

$$P(max_{1\le k\le n}|S_k| > \epsilon) \le \sum_{i=1}^{n} \frac{\sigma_i^2}{\epsilon^2}.$$

(4) Let T_ν be a t random variable with ν degrees of freedom, with the following density function

$$f_T(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}(1 + t^2/\nu)^{(\nu+1)/2}}, -\infty < t < \infty.$$

Prove that the density function of T_ν can be written as as the following mixture of normals:

$$f_{T_\nu}(t) = \int_0^\infty \phi(t\sqrt{x}) w(\nu x) dx$$

where $\phi(\cdot)$ is the probability density function of $N(0, 1)$, and

$$w(s) = \frac{\sqrt{x}}{\Gamma(\nu/2)2^{\nu/2}} s^{(\nu/2)-1} e^{-s/2}.$$

Verify this formula by direct integration and by conditional probability.

(5) Let $\phi(y)$ be a positive, even, and continuous function on $(-\infty, \infty)$ such that $\phi(y)$ is strictly decreasing on $(0, \infty)$, and $\int_{-\infty}^{\infty} \phi(y) dy = 1$. Consider the following bivariate density function:

$$f(x, y) = \begin{cases} 1 + x/\phi(y), & \text{if } -\phi(y) \le x < 0 \\ 1 - x/\phi(y), & \text{if } 0 \le x \le \phi(y) \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(x, y)$ be the corresponding cumulative distribution function,

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t) dt ds.$$

Show that if $0 < \Delta x < \phi(0)$, then

$$F(\Delta x, 0) - F(0, 0) \ge \int_0^{\phi^{-1}(\Delta x)} \int_0^{\Delta x} [1 - \frac{s}{\phi(t)}] ds dt \ge \frac{1}{2} \Delta x \phi^{-1}(\Delta x)$$

where ϕ^{-1} is the inverse function of $\phi(y)$ for $0 \le y < \infty$. Also show that $\partial F(x, y)/\partial x$ does not exist at (0,0).

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科目：數值分析【應數系乙組】

Twenty points for each problem. Please write down all the detail of your computation and proof.

1. Let γ be a simple root of nonlinear equation $f(x) = 0$ where $f(x)$ is smooth. Apply the Newton method to find γ. Show that this Newton iteration converges quadratically if the initial guess is sufficiently close to γ. What happens if γ is a double root?

2. (1) Find the linear polynomial $p(x)$ on $[0,1]$ such that $\|p(x) - \sqrt[3]{x}\|_\infty = \sup_{x\in[0,1]} |p(x) - \sqrt[3]{x}|$ is minimal.

   (2) Find the quadratic polynomial $q(x)$ on $[-1,1]$ such that $\|q(x) - e^x\|_2 = (\int_{-1}^{1} |q(x) - e^x|^2 dx)^{\frac{1}{2}}$ is minimal.

3. Let A be a rank deficient $m \times n$ matrix with $m \geq n$. How to use (1) QR with column pivoting, (2) singular value decomposition to obtain the least squares solution of $A\mathbf{x} = \mathbf{b}$? Explain why they work?

4. Let T be an $n \times n$ matrix and $\mathbf{v}$ be an n dimensional column vector. Prove that the iterative method $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{v}$ converges if, and only if, the spectral radius $\rho(T) < 1$. Please provide the detail for all theorems you use in the proof.

5. (1) Derive the Euler's method to solve the initial value problem of ODEs

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a,b] \\ y(a) = y_0 \end{cases}$$

   with local truncation error.

   (2) Apply the Euler method to solve

$$\begin{cases} y'(t) = \sqrt{y(t)}, & t \in [0,1] \\ y(0) = 0 \end{cases}$$

   and compare the numerical solution to the exact solution. What goes wrong?

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科目：分 析 【應數系丙組選考】

1. A subset A of a topological space is said to be *nowhere dense* if the closure of A has empty interior. (30 %)

   (a) Show that a set A is nowhere dense if and only if every non-empty open set has a non-empty open subset disjoint from A.

   (b) Show that a closed set is nowhere dense if and only if its complement is everywhere dense.

   (c) Show that the boundary of a closed set is nowhere dense.

2. Show that a compact metric space is separable. (10%)

3. Use residues to find the following integral (20%)

$$\int_{-\infty}^{\infty} \frac{\cos 2x dx}{x^2 + 2x + 2}.$$

4. Show that for any $m \in N$, the sequence $\{(1+\frac{m}{k})^k : k \in N\}$ is increasing and bounded above. Show that the sequence converges to a real number $\psi(m)$ and show that $\psi(m) = (\psi(1))^m$. (20 %)

5. Let H be a real Hilbert space and K be a closed convex cone in H. Let $T : K \to H$ and onsider the following two problems:

   (VI) Find $x \in K$ such that $\langle Tx, y - x \rangle \geq 0 \ \ \forall y \in K$.

   (CP) Find $x \in K$ such that $\langle Tx, x \rangle = 0$ and $Tx \in K^*$ where $K^* = \{v \in H : \langle v, y \rangle \geq 0 \ \ \forall y \in K\}$.

   Show that x solves (VI) if and only if x solves (CP). (20%)

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科目：組合數學【應數系丙組選考】

1. Let $O(G)$ be the set of all orientations of the graph G and $l(D)$ be the length of a longest directed path in the digraph D. Prove that the chromatic number of G is $\min\{l(D): D\in O(G)\}+1$. (20%)

2. Suppose G is a bipartite graph with the maximum degree k. Prove that the chromatic index of G is k. (20%)

3. Suppose G be a graph with n vertices, $(n-1)(n-2)/2+2$ edges and $n\geq 3$. Prove that G has a Hamiltonian cycle. (20%)

4. Let $\chi(G,k)$ be the chromatic polynomial of the graph G.
   (a) Find $\chi(C_n,k)$ where C_n is a cycle with n vertices. (10%)
   (b) Let G be an outerplane graph with n vertices and each region of G be either a triangle or a Hamiltonian cycle. Determine $\chi(G,k)$. (10%)

5. Prove that, for each integer $k\geq 3$, there exists a graph G with the chromatic number at least k and the girth at least 4. (20%)