25 points for each of the following problems

1. Let A_n be the square $\{(x, y) : |x| \leq 1, |y| \leq 1\}$ rotated through the angle $2\pi n\theta$, $n = 1, 2, ...$ Explain what limsup A_n and liminf A_n are in case
(a) $\theta = 1/8$.
(b) θ is rational.
(c) θ is irrational.

2. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and f_n are measurable functions with respect to this space.
(a) Prove that if $0 \leq f_n \to f$ almost everywhere and $\int f_n d\mu \leq A < \infty$, then f is integrable and $\int f_n d\mu \leq A$.
(b) Suppose that f_n are integrable and $sup_n \int f_n d\mu < \infty$. Show that, if $f_n \uparrow f$, then f is integrable and $\int f_n d\mu \to \int f d\mu$.

3. Prove that if X_1,X_2,... are independent random variables (but need not be identically distributed), $E(X_n) = 0$, and$E(X_n^4)$ is bounded, then $S_n/n \to 0$ with probability 1, where $S_n = X_1 + X_2 + ... + X_n$.

4. State and prove the Central Limit Theorem for the i.i.d. case.

1. (20%) The random variable is said to have a Pareto distribution with parameter $a, b\,(a > 0, b > 0)$ if its density is

$$f(x; a, b) = \frac{a}{b(1 + x/b)^{a+1}}, \ x > 0,$$

where $F(x; a, b)$ denote the corresponding cumulatve distribution function (CDF). Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) with Pareto distribution $f(x; 1, 1)$.

   (i) Find the limiting distribution of random variable $V_n = \max(X_1, \ldots, X_n)/n$.

   (ii) Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the order statistics of $\{X_i, 1 \le i \le n\}$.

      (a) Prove that $F(X_1; a, b)$ has $U[0, 1]$ distribution.

      (b) Find the joint probability density function of $(F(X_{(1)}; a, b), F(X_{(n)}; a, b))$.

2. (20%) Let $X_1, \ldots, X_n$ be a random sample from the distribution with probability density function (p.d.f.)

$$f(x|\theta) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \ x \ge \mu, \theta = (\mu, \sigma), \mu \in R, \sigma > 0.$$

   (i) Find the maximum likelihood estimate(MLE) of μ and σ.

   (ii) Find the MLE of $P(X \ge t)$, where $t > \mu$.

3. (30%) Let $X_1, \ldots, X_n$ be a random sample from the distribution uniform on the interval $[0, \theta]$.

   (i) Find the sufficient statistics for θ, denote it as T_1, and find $E(T_1)$.

   (ii) Show that T_1 is a consistent estimate of θ.

   (iii) Let mean square error

$$MSE_\theta(T) = E_\theta[T(X_1, \ldots, X_n) - \theta]^2$$

   be the risk function of an estimator T and admissibility is defined with respect to the mean square error. Is T_1 an admissible estimator for θ? Prove or disprove it.

   (iv) Consider the estimator $T_3 = (n + 1)\min(X_1, \ldots, X_n)$. Show that T_3 is an unbiased estimator of θ and find an UMVUE estimator through T_3, and explain it.

4. (30%) Let $X_1, \ldots, X_n$ be a random sample from the distribution $N(\mu, \sigma^2)$ with known σ^2.

   (i) Consider the situation of testing the simple null hypothesis against the composite alternative hypothesis

   $$H_0 : \mu = 0, \text{ v.s. } H_1 : \mu > 0.$$

   Find an uniformly most powerful test for testing H_0 against H_1 on significance level α.

   (ii) Consider the situation of testing the composite null hypothesis against the composite alternative hypothesis

   $$H_0 : \mu \leq 0, \text{ v.s. } H_1 : \mu > 0.$$

   Find an uniformly most powerful test for testing H_0 against H_1 on significance level α, if there is any. If there isn't any explain why.

   (iii) Consider the situation of testing the simple null hypothesis against the composite alternative hypothesis

   $$H_0 : \mu = 0, \text{ v.s. } H_1 : \mu \neq 0.$$

   Find an uniformly most powerful test for testing H_0 against H_1 on significance level α, if there is any. If there isn't any explain why.

   (iv) Discuss the above three testing problems when σ^2 is unknown.

國立中山大學九十一學年度博士班招生考試試題

科目：數值分析【應數系乙組】

Entrance Exam for the Ph.D Program of Scientific Computing

Six questions with the marks indicated.

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1. (15) Describe convergence and stability for numerical methods, give relations between them, and provide examples to explain your answers.

2. (15) Given 26 English Capital letters, $A, B, ..., Z$, in standard forms. Describe the methods of recognizing a new character to be one of those 26 letters.

3. (15) Describe the singular value decomposition for the matrix $A \in R^{m\times n}, m \geq n$, and prove that the singular values are non-negative.

4. (15) Derive the error bounds for the trapezoidal rule in two dimensions:

$$\int_0^h \int_0^k g(x,y)dxdy \approx \frac{hk}{4}(g(0,0)+g(h,0)+g(0,k)+g(h,k)).$$

5. (20) Consider the system of ordinary differential equations,

$$\frac{d\vec{u}}{dt} + A\vec{u} = \vec{f},$$

where $\vec{f}$ is known and $\vec{u}(0)$ is given. The matrix A is symmetric and positive definite. Provide truncation errors, and derive stability analysis for the following scheme:

$$\frac{w^{n+1} - w^n}{\Delta t} + \frac{1}{2}A\{w^{n+1} + w^n\} = \vec{f}^{n+\frac{1}{2}}, \quad n \geq 0,$$

where w^n is used to approximate $\vec{u}$ at $n\Delta t$.

6.(20) Given an original image $\{\phi_{ij}\}$ and other two images, $\{u_{ij}\}$ and $\{v_{ij}\}$, consisting of 256×256 pixels with 256 greyness levels. Form a linear combination

$$\{w_{ij}\} = \alpha\{u_{ij}\} + \beta\{v_{ij}\}.$$

Provide a numerical method to seek the parameters α and β such that the combined image $\{w_{ij}\}$ is best approximate to the original image $\{\phi_{ij}\}$.

P.S. In Questions 3-5, suppose that the solution is smooth enough.

國立中山大學九十一學年度博士班招生考試試題

科目：分析【應數系丙組】

Answer all the following problems. Here $\mathbf{R}$ and $\mathbf{C}$ denote the sets of real and complex numbers respectively.

1. (10%) Show that a compact set $K \subset \mathbf{R}^n$ has to be closed and bounded.

2. (10%) Show that if ϕ is a continuous real function on (a,b) such that for all $x, y \in (a,b)$,
$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y) ,$$
then ϕ is convex on (a,b).

3. (10%) Let $f \in L^1(\mathbf{R})$, $g(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} dt$. Show that g is continuous at $x = 0$.

4. (10%) Find a complex mapping that maps the set $\{z \in \mathbf{C} : \ |z| < 1, \ \text{Re}\, z > 0, \ \text{Im}\, z > 0\}$ onto the half-plane $\text{Re}\, z > 1$.

5. (15%) Show that if (a_n) is a sequence in $\mathbf{R}$, and $\lim_{n\to\infty} a_n = a \in \mathbf{R}$, then
$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = a .$$

6. (15%) Let μ and ν be two positive measure on $\mathbf{R}$. Show that if μ is absolutely continuous with respect to ν, and the two measures are mutually singular, then $\mu = 0$.

7. (15%) Let H be a Hilbert space with inner product $<,>$ and norm $\|f\| = \sqrt{<f,f>}$. Suppose $M = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$ be the subspace spanned by $\phi_1, \ldots, \phi_n \in H$. Show that for any $f \in H$, there exists a unique $h \in M$ such that
$$\|f - h\| = \min_{g \in M} \|f - g\| .$$

8. (15%) Let Ω be an open subset of $\mathbf{C}$, and $f : \ \Omega \to \mathbf{C}$ is differentiable when Ω and $\mathbf{C}$ are viewed as sets in $\mathbf{R}^2$. Define
$$\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) , \qquad \overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) .$$
Show that f is holomorphic on Ω if and only if $\overline{\partial} f(z) = 0$ for all $z \in \Omega$.

國立中山大學九十一學年度博士班招生考試試題

科目：組合數學【應數系丙組】

1 (20 points). Prove that if G is a finite graph which is not complete and $\Delta(G) \geq 3$, then $\chi(G) \leq \Delta$.

2 (20 points). Without using Lovasz perfect graph theorem, prove that $\chi(G) = \omega(G)$ if $\bar{G}$ is a bipartite graph.

3 (20 points). State and prove Euler's formula for planar graphs.

4 (20 points). Suppose G is a bipartite graph. Prove that if λ is an eigenvalue of the adjacent matrix A of G of multiplicity m, then $-\lambda$ is also an eigenvalue of A of multiplicity m.

5 (20 points). Suppose $m > 2n$ are positive integers. An n-subset A of the set $\{1, 2, \cdots, m\}$ is called *stable* if for any two elements i, j of A, $2 \leq |i - j| \leq m - 2$. What is the total number of stable n-subsets of $\{1, 2, \cdots, m\}$?