Each of the following five problems will be given 20 points,

(一) Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space. δ is a non-negative measurable function. Answer the following questions:

(i) Define $\nu(A)=\int_A \delta\, d\mu$, $A \in \mathcal{F}$. Prove that ν is a measure.

(ii) If f is a non-negative measurable function. Prove that

$$\int f\, d\nu = \int f \delta\, d\mu$$

(iii) If f is integrable (not necessarily non-negative) with respect to ν, then $f\delta$ is integrable with respect to μ.

(二) Suppose $(\Omega, \mathcal{F})$ is a measurable space. $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$.

(i) If P is a probability measure then

$$\overline{\lim}\, P\{A_n\} \le P\left\{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right\} = P\left\{\overline{\lim}\, A_n\right\}.$$

(ii) If μ is an arbitrary measure then

$$\overline{\lim}\, \mu(A_n) \le \mu\{\overline{\lim}\, A_n\} \text{ may not be true.}$$

Give a counter example.

(三) Prove the following different version of Strong Law of Large Numbers. Random variables are defined on $(\Omega, \mathcal{F}, P)$.

(i) $X_1, X_2, \cdots, X_n$ are i.i.d r.v's with $E(X_i^4)<\infty$. $E(X_i)=m$. Let $S_n=\sum_1^n X_i$. Then $S_n/n \to m$ with probability one.

(ii) If we only assume that $E(X_i^2)<\infty$, the result in (i) is still true.

(iii) If $X_1, X_2, \cdots, X_n$ are i.i.d simple random variables (i.e. X_i only takes values in $x_1, x_2, \cdots, x_k$). If $E(X_i) = m$. Then

$$S_n/n \to m \text{ with probability one.}$$

(四). Let $\Omega = (0,1]$. Any $w \in \Omega$ can be expressed as a dyadic expansion, for example, $w = 0.875$ can be expressed as

$$w = 0.11100\cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 0 + \cdots = 0.875$$

In general, $w = 0.d_1 d_2 \cdots$, where $d_i = 0$ or 1, means $w = \sum_{k=1}^{\infty} \frac{d_k}{2^k}$.

Let $\mathcal{F}$ be the Borel σ-field of Ω and let P be the Lebesgue measure.

Define $l_n(w) = k$ iff $d_n(w) = d_{n+1}(w) = \cdots = d_{n+k-1}(w) = 0$ but $d_{n+k}(w) = 1$.

Prove the following result by Borel Cantelli Lemma.

(i) Let $A_n = \{w \in \Omega : l_n(w) = 1\}$. Then $P\{w: A_n \text{ infinitely often}\} = 1$.

(ii) Let $B_n = \{w \in \Omega : l_n(w) \geq (1+\varepsilon)\log_2 n\}$, $\varepsilon > 0$.

Then $P\{w: B_n \text{ infinitely often}\} = 0$.

(五) Suppose $X_1, X_2, \cdots$ and X are random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, f is a continuous function.

Prove the following result:

(i) If X_n converges to X a.e then $f(X_n)$ converges to $f(X)$ a.e.

(ii) If X_n converges to X in probability then $f(X_n)$ converges to $f(X)$ in probability.

1. Let Y_1, Y_2 and Y_3 be independent exponential(λ_i)

$$f(y_i) = \lambda_i \exp(-\lambda_i y_i), \qquad y_i \geq 0; \lambda_i > 0; i = 1, 2, 3$$

and $Y_{X_1} > Y_{X_2} > Y_{X_3}$ where (X_1, X_2, X_3) is a random vector of a permutation of $1, 2$ and 3. Show that (20%)

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{\lambda_{x_3}}{\lambda_{x_1} + \lambda_{x_2} + \lambda_{x_3}} \times \frac{\lambda_{x_2}}{\lambda_{x_1} + \lambda_{x_2}}$$

2. Let X be a random variable having a continuous CDF $F(x)$.

   (a) Show that if $X \geq 0$ a.s., then (10%)

$$EX = \int_0^\infty (1 - F(x))\, dx.$$

   (b) Show that in general, if EX exists, then (10%)

$$EX = \int_0^\infty (1 - F(x))\, dx - \int_{-\infty}^0 F(x)\, dx.$$

3. Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, \theta^2)$ distribution, where $\theta > 0$ is a parameter. Find a minimal sufficient statistic for θ and show whether it is complete. (20%)

4. Let $X_1, \ldots, X_n$ be i.i.d. having log distribution

$$f(x) = -(\log p)^{-1} x^{-1} (1-p)^x, \qquad x = 1, 2, \ldots; 0 < p < 1.$$

   (a) For $n = 2$, find the UMVUE of p^k. (10%)

   (b) For $n = 3$, find the UMVUE of $P(X = k)$. (10%)

5. Let $X_1, \ldots, X_n$ be i.i.d. from

$$f_\theta(x) = \theta^{-1} \exp(-x/\theta), \qquad x \geq 0; \theta > 0.$$

Find a UMP test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. (20%)

國立中山大學九十學年度博士班招生考試試題

科目：**數值分析【應數所乙組】**

Entrance Exam for the Ph.D Program of Scientific Computing

Six questions with the marks indicated.

1.(15) A double root x^* of the nonlinear equation $f(x) = 0$ is given by $f(x^*) = 0$, $f'(x^*) = 0$ and $f''(x^*) \neq 0$. Suppose that the real computation is carried out by $f(\tilde{x}) + \delta g(\tilde{x}) = 0$, where $g(\tilde{x}) = O(1)$, and δ is the rounding errors of computer. Give stability analysis for the approximate solution $\tilde{x}$.

2. (15) For the integral $\int_0^1 f(x)dx$, write down the composite Simpson rule of integration and its error order with respect to the maximal section length h. Moreover, give a reasonable explanation for such an error order.

3.(15) Consider the initial value problem of ordinary differential equations (ODEs),

$$y' = f(x, y), x \geq 0;\ y(0) = y_0.$$

Give the midpoint scheme, derive the local errors and provide stability analysis.

4.(15) Choose iteration methods to solve the nonsingular linear algebraic equations: $Ax = b$, where $A \in R^{n\times n}$, $x \in R^n$, $b \in R^n$. Let the matrix A be split as $A = L + T$ with $L \in R^n$ and $T \in R^n$, and construct the iteration method:

$$x_{k+1} = x_k - (Lx_{k+1} + Tx_k + b),\ k \geq 0,$$

where x_0 is an initial vector.
(1) Give the sufficient and necessary condition of convergence of the given iteration method.
(2) Prove them.

5. (20) Describe a method for finding the minimal eigenvalue of matrix $B^{-1}A$, where matrices A and B are symmetric and positive definite. Moreover, provide error analysis.

6.(20) Consider the Poisson equation:

$$-\{\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\} = f(x,y)\ \ in\ \ S,$$

with the mixed type of the boundary conditions:

$$u\Big|_{\Gamma_D} = g_1,\ \{-\frac{\partial u}{\partial n} + \alpha u\}\Big|_{\Gamma_R} = g_2,$$

where n is the outnormal to ∂S, $\alpha \geq 0$ and $\partial S = \Gamma_D \cup \Gamma_R$.
(1) Describe the linear finite element method(FEM),
(2) Give briefly error analysis.

Problem 1 carries 15 points and Problems 2-6 carry 17 points each.

(1) Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions on a measure space $(\mathbf{X}, \mathcal{F}, \mu)$. Show that

$$\int_{\mathbf{X}} \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_{\mathbf{X}} f_n \, d\mu.$$

(2) Let $\{\mu_n\}$ be a sequence of measures on the measurable spaces $(\mathbf{X}, \mathcal{F})$. Sup--pose that there is a set function μ such that $\lim_{n\to\infty} \mu_n(E) = \mu(E)$ for all $E \in \mathcal{F}$. Prove or disprove: μ is a measure on $\mathcal{F}$.

(3) Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Show that $\mathbf{X}$ is a Banach space if and only if every series $\{\mathbf{x}_n\}$ in $\mathbf{X}$ with $\sum_{n=1}^{\infty} \|\mathbf{x}_n\| < \infty$ converges.

(4) Let f be holomorphic on a connected open set Ω, and $Z(f) = \{z \in \Omega : f(z) = 0\}$. Show that $Z(f) = \Omega$, or $Z(f)$ has no limit point.

(5) If $1 < p < \infty$, prove or disprove: The unit ball of $L^p(\mathbf{R}, \mu)$ is *strictly convex*, i.e., if $\|f\|_p = \|g\|_p = 1$, $f \neq g$, $h = \frac{1}{2}(f+g)$, then $\|h\|_p < 1$. (The surface of the unit ball contains no straight lines.)

(6) Suppose that $f \in L^1([-\pi, \pi])$ and let, $z = re^{i\theta}$,

$$P[f](z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta, t) f(t) \, dt,$$

where $P_r(\theta, t)$ is the Poisson kernel

$$P_r(\theta, t) = \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}.$$

Show that $P[f]$ is harmonic in the unit disc.

1 (15 points). Prove that any graph G has a subgraph H which is bipartite and $|E(H)| \geq |E(G)|/2$.

2 (20 points). (a) Prove that if G has n vertex and more than $n^2/4$ edges, then G has a vertex x such that $G - x$ has more than $(n-1)^2/4$ edges.

(b) Use the above result, prove by induction that if an n vertex graph G has more than $n^2/4$ edges then G contains a triangle.

3 (15 points). Prove that every tournament (an oriented graph in which there is exactly one directed edge between any pair of distinct vertices) contains a directed Hamilton path.

4 (10 points).

$$x_1 + x_2 + x_3 + x_4 + x_5 < 100, x_1 \geq -1, x_2 \geq -2, x_3 \geq -3, x_4 \geq -4, x_5 \geq -5.$$

5 (15 points). State and prove Euler's formula for connected plane graphs.

6 (15 points). Prove that every K_3-free planar graph is 4-colorable.

7 (10 points). Prove that a matching M of a graph G is maximum if and only if there is no augmenting path with respect to M.