國立中山大學八十八學年度碩博士班招生考試試題

科　目：應數系機率論（應用數學系博士班）　共 1 頁 第 1 頁

Do *all* the problems in detail.

(1) If the independent random variables X and Y satisfy [15%]

$$P\{|X_Y| \geq \epsilon\} \leq \epsilon$$

for some ϵ, then their distribution functions F and G satisfying the inequalities:

$$F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon.$$

(2) State without proof the Kolmogorov's extension theorem. [15%]

(3) Prove or disprove: $[E(|X|^p)]^{1/p} \geq [E(|X|^q)]^{1/q}$ for $1 < p < q < \infty$. [15%]

(4) [20%]
(a) Prove or disprove: Convergence almost surely implies convergence in L^1.
(b) Prove or disprove: Convergence in L^1 implies convergence in probability.

(5) Let X be a random variable with $Pr(X=-1) = \frac{1}{2}$ and $Pr(X=1) = \frac{1}{2}$, and $X_n = (-1)^n X$. Show that X_n converges to X in distribution, but not in probability. [15%]

(6) Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of independent random variables with

$$Pr(X_n = 1) = Pr(X_n = -1) = \frac{1}{2},$$

$$Pr(Y_n = 1) = Pr(Y_n = -1) = \frac{1}{2}(1 - \frac{1}{n^2}),$$

and

$$Pr(Y_n = \sqrt{n+1}) = Pr(Y_n = -\sqrt{n+1}) = \frac{1}{2n^2}.$$

Let $S_n = \sum_{k=1}^n X_k$ amd $T_n = \sum_{k=1}^n Y_k$. Prove or disprove the following:

(i) S_n/n converges to zero a.s.
(ii) T_n/n converges to zero a.s.
(iii) $S_n/\sqrt{n}$ converges to $N(0,1)$ in distribution.
(iv) $T_n/\sqrt{n}$ converges to $N(0,1)$ in distribution. [20%]

1. 設 $U_1, U_2, \cdots,$ 為 i.i.d Uniform $(0,1)$.
X 為另一個與 $U_1, U_2, \cdots$ 独立的 random variable. $P(X=k)=\frac{c}{k!}$, $k=1,2,\cdots$
$c=\frac{1}{e-1}$. 回答以下二小題.

(a) 求 $W=\min(U_1, U_2, \cdots U_n)$ 的分佈. 20分

(b) 求 $Z=\min(U_1, U_2, \cdots, U_X)$ 的分佈.

2. 請舉一反例說明：X 和 Y 的相關係數 $\rho(X,Y)=0$ 並不一定有 X, Y 互為独立的結果。 20分

3. 一盒內有 10 球，其中白色的有 M 個，剩下的為黑色的.
從盒中任取三球（不放回）. 要 test

H_0: $M=5$
H_1: $M=6$ 20分

請回答以下問題

(a) 設 type I error $\alpha=0.1$. 則 most powerful test 為何？

(b) 試計算其 power.

4. 設 $X_1, X_2, \cdots, X_n$ iid 其 p.d.f 為 $f(x;\theta)=\theta x^{\theta-1}$, $x\in(0,1)$, $\theta>0$.
令 $g(\theta)=\frac{1}{\theta}$, 請找一個 $g(\theta)$ 的 unbiased estimator 且其 variance 又達到 Cramer-Rao 的 lower bound. 20分

5. 設 $X_1, X_2, \cdots$ 為 iid $P(X_i = \pm 1) = \frac{1}{2}$. 令

$$Z = \sum_{k=1}^{\infty} \frac{X_k}{2^k}.$$

20分

用 Moment Generating Function 証明 Z 為 $U(-1,1)$.

科　目：**數值分析**（應用數學系博士班）　

每題20分，無計算或證明過程者，不予計分。禁止使用計算器。

1. ① State and prove the Aitken's Δ^2 process to accelerate linearly convergent sequence.

   ② Show that Aitken's Δ^2 sequence converges to the limit faster than original sequence.

2. ① Write down Lagrange formula and Newton divided differences formula for polynomial interpolating $f(x)$ at $x_0, x_1, \cdots, x_n$. What's error?

   ② Derive Trapezoidal Rule for numerical integration with error formula.

3. State how to use ① normal equation, ② QR factorization, ③ singular value decomposition to solve discrete linear least squares problem. Sketch the proofs.

4. Prove the following are equivalent for $n \times n$ matrix A:

   ① for all (i,j), the (i,j) entry of A^k converges to 0 as $k \to \infty$,

   ② $\lim_{k \to 0} \|A^k\| = 0$ for all matrix norms $\|\cdot\|$,

   ③ Spectral radius $\rho(A) < 1$,　⑤ $\lim_{k \to \infty} A^k x = 0$, $\forall$ n dimensional vector x.

5. ① Derive Euler's method to solve $\begin{cases} y'(t) = f(t, y(t)), & t \in [a,b], \\ y(a) = \alpha, \end{cases}$ with local truncation error.

   ② Apply Euler's method to solve $\begin{cases} y'(t) = \sqrt{y(t)}, & t \in [0,1], \\ y(0) = 0, \end{cases}$ and compare the numerical solution to exact solution. What goes wrong?

Problems 1—6 carry 10 points each and Problems 7 and 8 carry 20 points each.

1. Let X be a compact space and Y be a Hausdorff space. Show that every one-to-one continuous mapping from X onto Y is a homeomorphism.
2. Let (X, d) be a metric space and $\{x_n\}$ be a Cauchy sequence in X. Suppose that $\{x_n\}$ has a convergent subsequence. Show that the sequence $\{x_n\}$ itself is convergent.
3. Given a bounded measurable function $f : [0,1] \longrightarrow \mathbb{R}$ and $\varepsilon > 0$. Find a continuous function $g : [0,1] \longrightarrow \mathbb{R}$ such that
$$m\{x \in [0,1] : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon.$$
4. Prove that if a subset A of $[0,1]$ has measure zero then
$$A^2 = \{x^2 : x \in A\}$$
has measure zero, too.
5. Show that the most general Möbius transformation from the upper half plane to the unit disk can be written in the form
$$w = e^{i\theta}\frac{z-\mu}{z-\overline{\mu}}.$$
6. Suppose that f is a real-valued differentiable function on $[0,1]$. Prove that its derivative function f' is Lebesque measurable on $[0,1]$.
7. Prove that if a real-valued function f is integrable on $[a,b]$ and
$$\int_a^x f(t)\,dt = 0$$
for all x in $[a,b]$ then $f(t) = 0$ a.e. in $[a,b]$.
8. Let μ_n be a non-decreasing sequence of measures defined on a measurable space $(X, \mathcal{A})$ in the sense that $\mu_n(A) \uparrow \mu(A)$ for all A in $\mathcal{A}$.
   (a) Prove that μ is a measure on $(X, \mathcal{A})$ with respect to which all μ_n are absolutely continuous.
   (b) On any fixed set A of finite μ measure, let f_n denote the Radon–Nikodym derivative of μ_n with respect to μ. Prove that almost everywhere (with respect to μ, and thus all μ_n) on A, $f_n \uparrow 1$.