100學年度博士班招生考試. 應用數學系甲組. 機率論

Do all problems in detail. 20 points for each problem.

1. Let X, Y be independent Poisson random variables with $E(X) = \lambda$ and $E(Y) = \mu$, and let $Z = X + Y$. Show that
$$E[X|Z=2] = \frac{2\lambda}{\lambda+\mu}.$$

2. Let X and Y be independent exponential random variables with mean $1/5$. Suppose $f(t) = t^3$ and $g(t) = e^t$ for all $t \in \mathbb{R}$.
   (a) Show that $[f(X) - f(Y)][g(X) - g(Y)] \geq 0$ with probability 1.
   (b) Show that $E[f(X)g(X)] \geq E[f(X)]E[g(X)]$.

3. Let X have a uniform distribution on $[1/2, 3/2]$. Show that $E[X \log X] \geq 0$.

4. Assume $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables with probability density functions $f(x) = 2x, 0 \leq x \leq 1$. For all $n \in \mathbb{N}$, let $Y_n = 1$ if $X_n < 1/2$ and $Y_n = 0$ if $X_n \geq 1/2$. Show that
$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i(1 - Y_j) \qquad \text{exists with probability 1}$$
and find the limit.

5. Let $\{A_n\}_{n\geq 1}$ be a sequence of events and $B_n = A_n^c \cap A_{n+1}$. Assume
$$\sum_{n=1}^{\infty} P(B_n) < \infty \qquad \text{and} \qquad \lim_{n\to\infty} P(A_n) = 0.$$
   (a) Show that $P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n\right) = 0$.
   (b) Show that $P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) = 0$.

~ End of Paper ~

每題 20 分，共 5 題，請詳列計算和推導過程書寫於題目下方空白處。

問題	1: 20 分	2: 20 分	3: 20 分	4: 20 分	5: 20 分	總分: 100 分
得分						

1. Let $(X_1, \ldots, X_n)$, $n > 2$, be a random sample from the uniform distribution on the interval $(\theta_1 - \theta_2, \theta_1 + \theta_2)$, where $\theta_1 \in \mathbb{R}$ and $\theta_2 > 0$. Find the UMVUE's of θ_j, $j = 1$, 2, and θ_1/θ_2.

2. Show that the priors in the following cases are conjugate priors:

   (a) $X = (X_1, \ldots, X_n)$ is a random sample from $N_k(\theta, I_k)$, $\theta \in \mathbb{R}^k$, and the prior is $N_k(\mu_0, \Sigma_0)$;

   (b) $X = (X_1, \ldots, X_n)$ is a random sample from the binomial distribution with probability θ and size k (a known positive integer), $\theta \in (0, 1)$, and the prior is the beta distribution with parameter (α, β).

3. Consider the estimation of an unknown parameter $\theta \in \mathbb{R}$ under the squared error loss. Show that if T and U are two estimators such that $P(\theta - t < T < \theta + t) \geq P(\theta - t < U < \theta + t)$ for any $t > 0$, then the squared error loss $R_T(P) \leq R_U(P)$.

4. In each of the following situations, calculate the p-value of the observed data.

   (a) For testing $H_0 : \theta \leq \frac{1}{2}$ versus $H_1 : \theta > \frac{1}{2}$, 7 successes are observed out of 10 Bernoulli trials.

   (b) For testing $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1, X = 3$ are observed, where $X \sim$ Poisson(λ).

5. Let X be a single observation from the beta$(\theta, 1)$ pdf.

   (a) Let $Y = -(\log X)^{-1}$. Evaluate the confidence coefficient of the set $[y/2, y]$.

   (b) Find a pivotal quantity and use it to set up a confidence interval having the same confidence coefficient as the interval in part (a).

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Ph.D. Entrance Examination
Numerical Analysis
May 6, 2011

(乙組)

Please write down all the detail of your computation and proof.
Twenty points for each problem.

1. Let γ be a simple root of nonlinear equation $f(x) = 0$ where $f(x)$ is smooth. Apply the Newton method to find γ. Show that this Newton iteration converges quadratically if the initial guess is sufficiently close to γ. What happens if γ is a double root?

2. Find (1) the Hermite polynomial, (2) the free cubic spline that interpolates $\frac{1}{x}$ at $x_0 = \frac{1}{2}$ and $x_1 = 1$.

3. Derive the trapezoidal rule and composite trapezoidal rule for numerical integration with error formula.

4. Let T be an $n \times n$ matrix and $\mathbf{v}$ be an n dimensional column vector. Prove that the iterative method $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{v}$ converges if, and only if, the spectral radius $\rho(T) < 1$. Please provide the detail for all theorems you use in the proof.

5. Use Taylor method of order n to solve the initial value problem of ODEs

$$\begin{cases} y'(x) = x - y(x) + 1, & x > 0 \\ y(0) = 1 \end{cases}$$

with step size h. Simplify the recurrence formula for numerical approximation of $y(kh)$, $k = 0, 1, 2, \cdots$.

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Ph.D. Entrance Examination – Analysis (May, 2011)

Answer all the problems below. The total is 100%.

1. (a) (5%) Define the term *limit infimum of a sequence* $\underline{\lim}_{n\to\infty} x_n$ in $\mathbf{R}$.

   (b) (10%) If $\underline{\lim}_{n\to\infty} x_n = \infty$, what can you tell about the sequence $\{x_n\}$? Support your argument.

   (c) (10%) A function f defined on (0,1) is said to be lower semicontinuous at $a \in (0,1)$ if for any sequence $x_n \to a$, we have $f(a) \le \underline{\lim}_{n\to\infty} f(x_n)$. Give an example of a lower semicontinuous but not continuous function on $(0,1)$. Verify your answer.

2. (20%) Let $\{x_n\}$ be a sequence in $\mathbf{R}$ such that $\lim\limits_{n\to\infty} x_n = a$.

   (a) Show that $\lim\limits_{n\to\infty} \dfrac{x_1 + x_2 + \cdots + x_n}{n} = a$.

   (b) Suppose that all x_i's and a are positive, prove that $\lim\limits_{n\to\infty} (x_1 x_2 \cdots x_n)^{1/n} = a$ too.

3. (20%) Define the functions f and g on $[-1,1]$ by $f(x) = x^{1/3}$ and

$$g(x) = \begin{cases} x^2 \cos(\frac{\pi}{2x}) & \text{if } x \in [-1,1] \setminus \{0\} \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Show that both f and g are absolutely continuous on $[-1,1]$.

   (b) Show that the composition $f \circ g$ is not absolutely continuous on $[-1,1]$.

   Hint: Consider the variation of f with respect to the partition P_n given by

$$P_n = \left\{-1, 0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

4. (a) (6%) State the Fatou's lemma.

   (b) (14%) Let $\{g_n\}$ be a sequence of integrable positive functions defined on a measurable set $E \subset \mathbf{R}$, which converges a.e. to an integrable function g. Let $\{f_n\}$ be a sequence of measurable functions on E such that $|f_n| \leq g_n$ and f_n converges to f a.e. $x \in E$. Using part (a) or otherwise, show that if $\int_E g = \lim_{n\to\infty} \int_E g_n$, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

5. (15%) Let Ω be a measurable set in $\mathbf{R}^n$. Suppose that $\{f_n\}$ converges *weakly* to f in $L^2(\Omega)$. That is, for any $g \in L^2(\Omega)$, $\lim_{n\to\infty} \int_\Omega g f_n = \int_\Omega g f$. Show that the $\{f_n\}$ converges (strongly) to f in $L^2(\Omega)$ if and only if $\lim_{n\to\infty} \|f_n\|_2 = \|f\|_2$.

End of Paper